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Asymptotic solutions to differential–difference equations

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Abstract. Asymptotic expansions are derived for Bessel functions starting from the differential–difference equations they satisfy. These are just the well known Green–Liouville expansions obtainable from either the pure differential or pure difference equations satisfied by Bessel functions. The Bessel functions are considered because of their well known properties but the method described should be applicable to many more general situations.

1. Introduction

The Bessel functions $J_n(x)$ and $Y_n(x)$ satisfy the differential equation

$$\frac{d^2}{dx^2}f_n(x)+\frac{1}{x}\frac{d}{dx}f_n(x)+\left(1-\frac{n^2}{x^2}\right)f_n(x) = 0 \tag{1}$$

where the order n is a parameter of the equation. These functions also satisfy the difference equation

$$f_{n+1}(x)+f_{n-1}(x) = 2\frac{n}{x}f_n(x) \tag{2}$$

where x is now a parameter of the equation. Both of these equations may be used to investigate the properties of Bessel functions and indeed identical expansions may be derived from (1) or (2) using WKB methods (Dingle and Morgan 1967a,b).

In addition to these 'pure' equations the Bessel functions satisfy 'mixed' equations, in particular,

$$\frac{\partial}{\partial x}f_n(x) = -\frac{n}{x}f_n(x)+f_{n-1}(x) \tag{3a}$$

$$2\frac{\partial}{\partial x}f_n(x) = f_{n-1}(x)-f_{n+1}(x) \tag{3b}$$

which are partial differential–difference equations. We will be concerned with deriving well known expansions from (3*a*) (type A) and (3*b*) (type B).

The reason for this work is that partial differential–difference equations can exist without any (known) corresponding differential or difference equation. An example is the Raman–Nath equation (Berry 1966) for the amplitude of the nth diffracted beam when light is scattered by ultrasonic sound:

$$2\frac{\partial}{\partial x}f_n(x) = f_{n-1}(x)-f_{n+1}(x)+\mathrm{i}\rho n^2 f_n(x). \tag{4}$$

An additional general example is the occupation number of a quantum level as a function of time when transitions are occurring to and from neighbouring quantum levels.

In contrast to the formal theory of differential–difference equations (Bellmann and Cooke 1963) we have adopted a rather empirical approach. The Bessel functions are well known and thoroughly investigated functions of mathematical physics and provide an excellent example for establishing the procedures necessary to obtain solutions of the WKB type. It is difficult to judge the general applicability of this work. Other standard functions such as the Hermite polynomials can be dealt with along the same lines but problems do arise, for example, with the identification of particular solutions to equation (4).

In § 2 we show how to obtain the first two terms in the Green–Liouville expansion for type A while § 3 is concerned with type B. In § 4 we discuss the applicability of the methods to equation (4) and general points concerning extension of this work.

2. Type A

It is natural to look for a solution to (3*a*) in the form

$$f_n(x) = \exp Z_n(x) \tag{5}$$

following the principles of the WKB method for differential or difference equations. Assuming that $Z_n(x)$ is regular and defined for complex n we may write†

$$Z_{n-1}(x) = Z_n(x) - \frac{\partial}{\partial n}Z_n(x) + \frac{1}{2!}\frac{\partial^2}{\partial n^2}Z_n(x) - \frac{1}{3!}\frac{\partial^3}{\partial n^3}Z_n(x)\ldots \tag{6}$$

and we know that when applying the WKB method to equation (2) derivatives of $Z_n(x)$ other than the first are assumed small in the lowest-order approximation. Correspondingly we drop $\partial^2 Z_n(x)/\partial n^2$ and so on, to obtain the approximate equation,

$$\frac{\partial}{\partial x}Z_n(x) = -\frac{n}{x} + \exp\left(-\frac{\partial}{\partial n}Z_n(x)\right); \tag{7}$$

a non-linear partial differential equation in the variables n and x.

2.1. *First-order approximation*

To solve (7) we will use Charpit's method (Piaggio 1958 p 162). We write

$$p \equiv \frac{\partial Z}{\partial x}, \qquad q \equiv \frac{\partial Z}{\partial n}$$

so that (7) is equivalent to

$$F(p, q, n, x, Z) \equiv p - \mathrm{e}^{-q} + \frac{n}{x} = 0. \tag{8}$$

The system of differential equations corresponding to (7) is simply:

$$\underset{\mathrm{I}}{\frac{\mathrm{d}x}{-1}} = \underset{\mathrm{II}}{\frac{\mathrm{d}n}{-\mathrm{e}^{-q}}} = \underset{\mathrm{III}}{\frac{\mathrm{d}Z}{-p-q\,\mathrm{e}^{-q}}} = \underset{\mathrm{IV}}{\frac{\mathrm{d}p}{-n/x^2}} = \underset{\mathrm{V}}{\frac{\mathrm{d}q}{1/x}}. \tag{9}$$

† In a particular physical problem n may be confined to integer values. The assumption that $f_n(x)$ is defined for general values of n yields sensible results for pure difference equations (Dingle and Morgan 1967a, b) and so we follow the same procedure here.

Taking I with V we have $\mathrm{d}x/x = -\mathrm{d}q$ or

$$\frac{\partial Z}{\partial n} \equiv q = -\ln x + C_1. \tag{10}$$

We can now eliminate q from (8) to give

$$\frac{\partial Z}{\partial x} \equiv p = -\frac{n}{x} + x\,\mathrm{e}^{-C_1}. \tag{11}$$

These two equations for p and q may now be integrated to find the *complete* integral, namely

$$Z_n(x) = -n \ln x + \frac{x^2}{2}\,\mathrm{e}^{-C_1} + \phi_1(n) \tag{12}$$

and

$$Z_n(x) = -n \ln x + C_1 n + \phi_2(x). \tag{13}$$

For (12) and (13) to be compatible we must choose

$$\phi_1(n) = C_1 n + C_2, \qquad \phi_2(x) = \frac{x^2}{2}\,\mathrm{e}^{-C_1} + C_2, \tag{14}$$

where C_2 is a second constant of integration. The complete integral is therefore

$$Z_n(x) = -n \ln x + \frac{x^2}{2}\,\mathrm{e}^{-C_1} + C_1 n + C_2. \tag{15}$$

We must now check whether there are *singular* solutions. This may be done by eliminating C_1 or C_2 from the equations

$$\frac{\partial Z}{\partial C_1} = 0, \qquad \frac{\partial Z}{\partial C_2} = 0. \tag{16}$$

In our case this gives

$$0 = -\frac{x^2}{2}\,\mathrm{e}^{-C_1} + n \tag{17a}$$

$$0 = 1 \tag{17b}$$

and (17*b*) indicates the absence of singular solutions.

The *general* integral is

$$Z_n(x) = -n \ln x + \frac{x^2}{2}\,\mathrm{e}^{-C_1} + C_1 n + \phi(C_1) \tag{18}$$

where ϕ is an arbitrary function of C_1 and $C_1 = C_1(n, x)$ is determined from

$$\frac{\partial Z}{\partial C_1} = 0 = -\frac{x^2}{2}\,\mathrm{e}^{-C_1} + n + \frac{\partial \phi(C_1)}{\partial C_1}. \tag{19}$$

Any function ϕ may be chosen and the resulting $Z_n(x)$ will satisfy (7) as may be verified by direct substitution.

Our task is now to find a subsidiary condition which will pick out the solutions which correspond to the Bessel functions. For simplicity we will consider the case when n is an integer and we will show that the condition

$$f_n(x) = (-1)^n f_{-n}(x) \tag{20}$$

performs this role†. Equation (20) is equivalent to the statement

$$Z_n(x) = Z_{-n}(x) + in\pi. \tag{21}$$

From

$$\frac{\partial Z_n(x)}{\partial x} = \frac{\partial Z_{-n}(x)}{\partial x}, \qquad \frac{\partial Z_n(x)}{\partial n} = \frac{\partial Z_{-n}(x)}{\partial n} + i\pi, \tag{22}$$

we obtain, with the use of (19),

$$-\frac{n}{x} + x\,e^{-C_1^+} = \frac{n}{x} + x\,e^{-C_1^-} \tag{23a}$$

$$-\ln x + C_1^+ = \ln x - C_1^- + i\pi. \tag{23b}$$

Here $C_1^+ \equiv C_1(+n, x)$ and $C_1^- \equiv C(-n, x)$. Eliminating C^- with the aid of equations (23*a*) and (23*b*) it is easily shown that

$$\exp(C_1^+) = \left[\frac{n}{x} \mp \left(\frac{n^2}{x^2} - 1\right)^{1/2}\right] x^{-1}. \tag{24}$$

The function ϕ may now be determined by noting that

$$\frac{\partial \phi}{\partial C_1} = -\tfrac{1}{2} e^{C_1} \tag{25}$$

whence

$$\phi(C_1) = -\tfrac{1}{2} e^{C_1}.$$

Substitution for C_1 and ϕ in (18) now readily gives the solutions

$$Z_n(x) = \pm \left\{ n \ln\left[\frac{n}{x} + \left(\frac{n^2}{x^2} - 1\right)^{1/2}\right] - (n^2 - x^2)^{1/2} \right\} \tag{26}$$

which is the familiar form for the first-order WKB approximation to the Bessel functions (Dingle and Morgan 1967a).

2.2. *Second-order approximation*

Equation (3*a*) is equivalent to

$$\frac{\partial Z_n(x)}{\partial x} = -\frac{n}{x} + \exp\left(-\frac{\partial Z_n(x)}{\partial n} + \frac{1}{2}\frac{\partial^2 Z_n(x)}{\partial n^2} - \frac{1}{6}\frac{\partial^3 Z_n(x)}{\partial n^3} \dots\right). \tag{27}$$

If we denote the solution of (7) by $Z_n^0(x)$ and write

$$Z_n(x) = Z_n^0(x) + Z_n^1(x) + Z_n^2(x) \dots \tag{28}$$

then the equation for $Z_n^1(x)$ is defined as

$$\frac{\partial Z_n^0(x)}{\partial x} + \frac{\partial Z_n^1(x)}{\partial x} = -\frac{n}{x} + \exp\left(-\frac{\partial Z_n^0(x)}{\partial n}\right)\left(1 - \frac{\partial Z_n^1(x)}{\partial n} + \frac{1}{2}\frac{\partial^2 Z_n^0(x)}{\partial n^2}\right) \tag{29}$$

where we have dropped all derivatives of Z^2, Z^3 and so on, higher derivatives than

† Note that the same condition applies to the Raman–Nath equation (Berry 1966).

$\partial^2 Z^0/\partial n^2$ and higher powers of this quantity, and higher powers and derivatives of $\partial Z^1/\partial n$.

Now writing $\partial Z^1/\partial x \equiv p$ and $\partial Z^1/\partial n \equiv q$ we have to solve the equation

$$p\exp\left(\frac{\partial Z^0}{\partial n}\right)+q-\frac{1}{2}\frac{\partial^2 Z_0}{\partial n^2}=0 \tag{30}$$

which is defined to be an equation of Lagrange's form (Piaggio 1958, p 147). We now seek two independent integrals of the subsidiary equations,

$$\underset{\mathrm{I}}{\frac{\mathrm{d}x}{\exp(\partial Z^0/\partial n)}}=\underset{\mathrm{II}}{\frac{\mathrm{d}n}{1}}=\underset{\mathrm{III}}{\frac{\mathrm{d}Z^1}{\frac{1}{2}\partial^2 Z^0/\partial n^2}}. \tag{31}$$

Combining I with II we have

$$\frac{\mathrm{d}n}{\mathrm{d}x}=\frac{n}{x}\mp\left(\frac{n^2}{x^2}-1\right)^{1/2} \tag{32}$$

where explicit use has been made of equation (26). Progress is now complicated by the need to solve (32) which we do by using Charpit's method†.

We denote $\mathrm{d}n/\mathrm{d}x$ by t so that (32) is equivalent to

$$F(t,x,n)\equiv t-\frac{n}{x}\pm\left(\frac{n^2}{x^2}-1\right)^{1/2}=0. \tag{33}$$

The subsidiary equations for (33) are

$$\frac{\mathrm{d}x}{-1}=\frac{\mathrm{d}n}{-t}=\frac{-x\,\mathrm{d}t}{t} \tag{34}$$

and using I and III together,

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}t}{t}, \tag{35}$$

or

$$t=C_1x. \tag{36}$$

Substitution back into (32) yields an equation for n in terms of x:

$$C_1x-\frac{n}{x}\pm\left(\frac{n^2}{x^2}-1\right)^{1/2}=0. \tag{37}$$

Care must be taken in squaring this equation. If we do square (37) then

$$n=\frac{C_1x^2}{2}+\frac{1}{2C_1} \tag{38}$$

but on resubstitution into (37) this does not appear to be a solution for the positive sign. This dilemma is resolved by writing (37) as

$$C_1x-\frac{n}{x}\pm\left|\left(\frac{n^2}{x^2}-1\right)^{1/2}\right|=0 \qquad \left(\frac{n^2}{x^2}>1\right) \tag{39a}$$

$$C_1x-\frac{n}{x}\pm\mathrm{i}\left|\left(1-\frac{n^2}{x^2}\right)^{1/2}\right|=0 \qquad \left(\frac{n^2}{x^2}<1\right). \tag{39b}$$

† This equation may also be solved by the substitution $n = Ux$.

We also note that

$$C_1 = -\frac{1}{x}\left[-\frac{n}{x} \pm \left(\frac{n^2}{x^2}-1\right)^{1/2}\right]. \tag{40}$$

Consider now the case when $n^2/x^2 > 1$ and when the sign in (39*a*) is positive. The solution (38) is valid provided $C_1 < 0$ and $C_1 < -1/x$ as may be ascertained by substitution.

We will follow through this particular situation and show that these constraints on C_1 resolve a difficulty with the sign at a later stage. Returning to II and III in (31) we have, choosing the positive sign in (26),

$$\frac{\mathrm{d}n}{2(n^2-x^2)^{1/2}} = \mathrm{d}Z^1 \qquad (n^2 > x^2) \tag{41}$$

or substituting for x^2 in terms of n and C_1 from (38) we have

$$\frac{\mathrm{d}n}{2(n^2+C_1^{-2}-2C_1^{-1}n)^{1/2}} = \mathrm{d}Z^1 = \frac{\mathrm{d}n}{2[(n-C_1^{-1})^2]^{1/2}} \tag{42}$$

and the dilemma arises as to whether the denominator of (42) should be written as $(n-C_1^{-1})$ or $(C_1^{-1}-n)$. This is resolved by noting that

$$(C_1^{-1}-n) = -n+x\left[\frac{n}{x}+\left(\frac{n^2}{x^2}-1\right)^{1/2}\right] = +(n^2-x^2)^{1/2} \tag{43}$$

which is positive as it should be. We can now integrate

$$\mathrm{d}Z^1 = \frac{\mathrm{d}n}{2(C_1^{-1}-n)} \tag{44}$$

to obtain

$$Z^1 = -\tfrac{1}{2}\ln(C_1^{-1}-n)+C_2 \tag{45}$$

or substituting for C_1 we have the *complete* integral

$$Z^1 = -\tfrac{1}{4}\ln(n^2-x^2)+C_2. \tag{46}$$

This, in fact, is easily shown to be valid if we choose the other sign in (39*a*). The general integral is simply

$$Z_n^1(x) = -\tfrac{1}{4}\ln(n^2-x^2)+a_1\left\{\frac{1}{x}\left[\frac{n}{x}\mp\left(\frac{n^2}{x^2}-1\right)^{1/2}\right]\right\}+\phi(a_1) \tag{47}$$

where $a_1(n, x)$ is determined by

$$0 = \frac{1}{x}\left[\frac{n}{x}\mp\left(\frac{n^2}{x^2}-1\right)^{1/2}\right]+\frac{\partial\phi(a_1)}{\partial a_1}; \tag{48}$$

ϕ being an arbitrary function. Alternatively we may write

$$Z_n^1(x) = -\tfrac{1}{4}\ln(n^2-x^2)+\psi\left\{\frac{1}{x}\left[\frac{n}{x}\mp\left(\frac{n^2}{x^2}-1\right)^{1/2}\right]\right\} \tag{49}$$

where ψ is an arbitrary function.

Since Z^0 satisfied the condition (21), we must have

$$\frac{\partial Z_n^1(x)}{\partial x} = \frac{\partial Z_{-n}^1(x)}{\partial x}, \qquad \frac{\partial Z_n^1(x)}{\partial n} = \frac{\partial Z_{-n}^1(x)}{\partial n}. \tag{50}$$

Substitution from (49) quickly shows that this is only possible if ψ is a constant independent of a_1. Hence

$$\exp(Z_n^1(x)) = (n^2 - x^2)^{-1/4} \tag{51}$$

and this is the standard second-order correction for Bessel functions.

3. Type B

The procedure follows that of § 2 but we consider this problem in detail because differences occur in determining the arbitrary functions.

We again make the substitution (5) and proceding in the same way as before we have to solve the approximate equation

$$\frac{\partial Z_n(x)}{\partial x} + \sinh \frac{\partial Z_n(x)}{\partial n} = 0. \tag{52}$$

It is completely straightforward to find the *general* integral using Charpit's method. The general integral is

$$Z_n(x) = -x \sinh C_1 + C_1 n + \phi(C_1) \tag{53}$$

where $C_1(n, x)$ is determined by

$$0 = -x \cosh C_1 + n + \frac{\partial \phi(C_1)}{\partial C_1}, \tag{54}$$

ϕ being an arbitrary function. We again find no singular integrals.

Before considering the function $\phi(C_1)$ which corresponds to the Bessel functions, it should be noted that if we have a solution to (3*b*) then we may generate other solutions simply by letting any operator, which commutes with $\partial/\partial x$ or $\partial/\partial n$, act on the particular solution. This is because x and n do not occur explicitly in equation (3*b*).

We will first apply equations (21) and (22) which give the following constraints on C_1 and ϕ:

$$C_1(n, x) = -C_1(-n, x) + i\pi, \qquad \phi(C_1) = \phi(-C_1 + i\pi). \tag{55}$$

Thus in contrast to the situation with type A, we are unable to uniquely define ϕ by the condition (20). However (55) does imply that

$$\phi(C_1) = F(\sinh C_1) \tag{56}$$

since $\sinh(C_1) = \sinh(-C_1 + i\pi)$ and we may write

$$\cosh C_1 = \frac{n}{x - \partial F(\sinh C_1)/\partial \sinh C_1}. \tag{57}$$

The function $F(\sinh C_1)$ which generates the Bessel functions is F = constant as may be readily verified but there are an infinity of other solutions satisfying equation (52).

A specific example is the choice $F = a \sinh C_1$ where a is a constant. This generates the solutions $J_n(x-a)$ and $Y_n(x-a)$. It is natural to look for a further condition relating the behaviour at x to that at $-x$.

From the symmetry properties of equation (3*b*) we may find solutions which satisfy

$$f_n(x) = \pm(-1)^n f_n(-x). \tag{58}$$

If we insist that

$$f_n(x) = (-1)^n f_n(-x) \tag{59}$$

then this imposes the additional constraints:

$$C_1(n, x) = C_1(n, -x) + \mathrm{i}\pi, \qquad \phi(C_1) = \phi(C_1 - \mathrm{i}\pi). \tag{60}$$

Hence $F(\sinh C_1)$ is limited to be an even function of $\sinh C_1$.

If we take the negative sign in (58) then we find $\phi(C_1)$ must satisfy

$$\phi(C_1) = \phi(C_1 - \mathrm{i}\pi) + \mathrm{i}\pi. \tag{61}$$

This is incompatible with (55) and (60) but compatible with the alternative subsidiary condition $f_n(x) = -(-1)^n f_{-n}(x)$ providing we choose

$$\phi(C_1) = C_1 + F(\sinh C_1). \tag{62}$$

It appears that a symmetry condition is not sufficient to identify the Bessel functions uniquely. For the moment we will simply take $F(\sinh C_1) =$ constant and proceed to the second approximation. The equation analogous to (29) is simply

$$\frac{\partial Z_n^1(x)}{\partial x} + \frac{\partial Z_n^1(x)}{\partial n} \cosh\left(\frac{\partial Z_n^0(x)}{\partial n}\right) + \frac{1}{2}\frac{\partial^2 Z_n^0(x)}{\partial n^2} \sinh\left(\frac{\partial Z_n^0(x)}{\partial n}\right) = 0 \tag{63}$$

or putting in the explicit form for $Z_n^0(x)$

$$x\frac{\partial Z_n^1(x)}{\partial x} + n\frac{\partial Z_n^1(x)}{\partial n} + \frac{1}{2} = 0. \tag{64}$$

This equation may easily be integrated to give the general integral

$$Z_n^1(x) = -\tfrac{1}{2}\ln x + \psi\left(\frac{n}{x}\right) \tag{65}$$

where ψ is an arbitrary function. If we consider the symmetry properties of (64) then we may insist on solutions satisfying

$$Z_n(x) = Z_{-n}(x), \qquad Z_n(x) = Z_n(-x). \tag{66}$$

These conditions only restrict ψ to be an even function of n/x. The Bessel functions correspond to choosing

$$\psi\left(\frac{n}{x}\right) = -\tfrac{1}{4}\ln\left(\frac{n^2}{x^2} - 1\right). \tag{67}$$

The unsatisfactory feature about type B equations is that the required arbitrary functions have not been deduced from first principles. This difficulty will now be discussed in § 4.

4. Other types of equation and boundary conditions

The origin of the logical difficulty encountered with identifying arbitrary functions for type B equations may be identified by considering a particular physical problem where a differential–difference equation occurs. The Raman–Nath equation (4) is ideal for this purpose since this equation reduces to the Bessel equation when the parameter ρ is small. The boundary condition (Berry 1966)

$$f_n(0) = \delta_{n,0} \tag{68}$$

is also satisfied by the Bessel function $J_n(x)$ and in fact a boundary condition of this type must be introduced to uniquely identify the solution to a type B equation. Differentiation of the Raman–Nath equation gives

$$\frac{\partial^r f_n(0)}{\partial x^r} = \frac{1}{2}\left(\mathrm{i}\rho n^2 \frac{\partial^{r-1} f_n(0)}{\partial x^{r-1}} + \frac{\partial^{r-1} f_{n-1}(0)}{\partial x^{r-1}} - \frac{\partial^{r-1} f_{n+1}(0)}{\partial x^{r-1}}\right) \tag{69}$$

so that all the derivatives of $f_n(x)$ at $x = 0$ are fixed by the condition (68). In particular if we consider the case when $n = r$ then from (69) we find that for $n \geqslant 0$

$$\frac{\partial^n f_n(0)}{\partial x^n} = (\tfrac{1}{2})^n. \tag{70}$$

This behaviour near $x = 0$ may be obtained if

$$f_n(x) \simeq \frac{1}{n!}\left(\frac{x}{2}\right)^n$$

for small x.

The boundary condition at $x = 0$ is inconvenient for determining arbitrary functions since the WKB expansion may break down at the boundary. This, for example, is the case for the Bessel functions when $n = x = 0$. The question arises as to whether or not a type B equation can be converted to a type A form.

The Raman–Nath equation may be expressed as

$$\frac{\partial f_n(x)/\partial x - f_{n-1}(x) - \frac{1}{2}\mathrm{i}\rho n^2 f_n(x)}{f_n(x)} = \frac{-\partial f_n(x)/\partial x - f_{n+1}(x) + \frac{1}{2}\mathrm{i}\rho n^2 f_n(x)}{f_n(x)} \tag{71}$$

or

$$\frac{\partial}{\partial x} f_n(x) - f_{n-1}(x) - \frac{\mathrm{i}\rho n^2}{2} f_n(x) = -\beta(n, x) f_n(x), \tag{72a}$$

$$\frac{\partial}{\partial x} f_n(x) + f_{n+1}(x) - \frac{\mathrm{i}\rho n^2}{2} f_n(x) = \beta(n, x) f_n(x). \tag{72b}$$

If we add (72*a*) and (72*b*) then we simply obtain equation (4) irrespective of the form of the unknown function $\beta(n, x)$. It should be noted that the symmetry condition (20) implies $\beta(n, x) = -\beta(-n, x)$.

The source of the ambiguity in identifying a particular solution to a type B equation is now apparent. We may insert any function $\beta(n, x)$ which is odd in the variable n, solve either (72*a*) or (72*b*) using the condition (20), and the resulting solution will satisfy the corresponding type B equation. This does not mean that the boundary condition (68) will be satisfied. Only if we choose a unique form for $\beta(n, x)$ will (68) be satisfied.

The problem of defining a particular solution to a type B equation then devolves into finding the form of $\beta(n, x)$.

One may formally solve the Raman–Nath equation or equation (3*b*) by writing ($n \geqslant 0$)

$$f_n(x) = \sum_{p=0}^{\infty} \frac{a_p(n) x^{n+p}}{(2)^n (n)!} \tag{73}$$

where $a_0(n) = 1$. The boundary condition (68) is satisfied by this form and the coefficients $a_p(n)$ may be determined by equating the coefficients of the separate powers of x to zero, after substitution into the type B differential–difference equation. In the case of the Bessel equation the general form of $a_p(n)$ can easily be discovered and one can then deduce the form $\beta(n, x) = n/x$ which will yield the same function $f_n(x)$ as given by (73).

This is rather an inelegant procedure and cannot easily be carried through for the Raman–Nath equation. In this case the coefficients $a_p(n)$ satisfy a more complicated recurrence relation and the general form cannot easily be deduced. We may write

$$\beta(n, x) = \frac{n}{x} \sum_{p=0}^{\infty} b_p(n) x^p \tag{74}$$

with $b_0(n) = 1$ and deduce the coefficients one at a time, so that, for example, we find

$$b_1(n) = -\frac{\mathrm{i}\rho}{12}(4|n| - 1). \tag{75}$$

The problem of determining $\beta(n, x)$ for the Raman–Nath equation is one which we have not been able to solve as yet. However, it should be remarked that the equations for the lowest-order approximation may readily be integrated to give the general integral

$$Z_n^0(x) = C_1 x + \int^n \sinh^{-1}\left(\frac{\mathrm{i}\rho n^2}{2} - C_1\right) \mathrm{d}n + \phi(C_1) \tag{76}$$

where $C_1 = C_1(n, x)$ is to be regarded as a constant in the integration over n and again ϕ is an aribtrary function. The fact that a type B equation requires more subsidiary information than a type A equation undoubtedly stems from the fact that type B involves a second-order difference operator.

There is a good deal to be discovered about the properties of differential–difference equations. If the difficulty about finding the function $\beta(n, x)$ in general, can be resolved then an obvious extension of this work would be to develop generalized WKB or uniform expansions along the lines described by Dingle and Morgan (1967b). Our main objective in this paper has been to show that differential–difference equations may be tackled directly and the kind of techniques which may be employed.

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